4.1 Eigenvalues and Eigenvectors (2D). $\mathbb{R}^2$

Consider the matrix A given by

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

and $\vec{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Notice that

a) $A\vec{u} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = (1)\vec{u}$

b) $A\vec{v} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4\vec{v}$.

c) $A\vec{w} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \end{bmatrix} \neq \lambda\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, for any λ in $\mathbb{R}$.

These computations suggest that the pairs:

$\lambda_1 = 1$ and $\vec{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and

$\lambda_2 = 4$ and $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

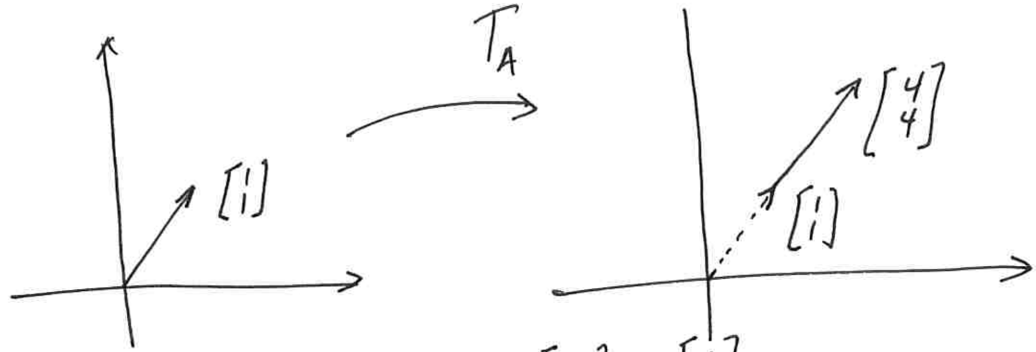
are special scalars and vectors. These expressions are particular cases of

$$A\vec{x} = \lambda\vec{x}$$

a) $\lambda = 1$, $\vec{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

b) $\lambda = 4$, $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Geometrical Meaning



$$A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}4\\4\end{bmatrix} = 4\begin{bmatrix}1\\1\end{bmatrix}$$

It means that the new vector

$$A\hat{x} = \lambda\hat{x}$$

is a multiple of the original vector $\vec{x}$.

So, it should be on the same line where $\hat{x}$ is lying.

Also notice that any multiple of $\begin{bmatrix}1\\1\end{bmatrix}$ has the same property. For example,

$$A\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}3 & 1\\2 & 2\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}8\\8\end{bmatrix} = 4\begin{bmatrix}2\\2\end{bmatrix}$$

The vectors $\overset{=\vec{v}}{\begin{bmatrix}1\\1\end{bmatrix}}$ and $\begin{bmatrix}1\\-2\end{bmatrix}=\vec{u}$ are so special that they deserve a special name. They are called <u>eigenvectors</u>. Moreover, the scalars $\lambda_1=1$ for $\begin{bmatrix}1\\-2\end{bmatrix}=\vec{u}$ and $\lambda_2=4$ for $\vec{v}=\begin{bmatrix}1\\1\end{bmatrix}$ are called eigenvalues of the matrix A.

<u>Natural Question:</u>

Given a matrix $A=\begin{bmatrix}3&1\\2&2\end{bmatrix}$ (for example)

How can we find its eigenvalues λ's and their corresponding eigenvectors?

$$A\vec{x}=\lambda\vec{x} \qquad (3.1)$$

So, find λ and $\vec{x}\neq\vec{0}$ satisfying (3.1).

This is not an easy question. The answer is based in <u>the invertibility theorem</u> for matrices.

Review this theorem with the matrix

$$A=\begin{bmatrix}1&2\\3&1\end{bmatrix}.$$

Exercise: Show that $\lambda = 4$ is an eigenvalue of

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$ by finding its corresponding eigenvectors.

It means there is $\vec{v} \neq \vec{0}$ Such that

$$A\vec{v} = 4\vec{v}.$$

To find $\vec{v}$

$$A\vec{v} - 4\vec{v} = 0 \quad \text{or} \quad (A - 4I\vec{v}) = 0$$

or $$\left(\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} 3-4 & 1 \\ 2 & 2-4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Row-reducing it

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 2 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow v_1 = v_2$$

$$\Rightarrow \vec{v} = \begin{bmatrix} v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ any multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector. for $\lambda = 4$

Finding the eigenvalues of A.

$$A\vec{x} = \lambda \vec{v} \qquad \underline{\underline{\lambda = ?}},\ \vec{v} = ?$$

or $$A\vec{x} - \lambda \vec{v} = \vec{0}$$

or $$A\vec{x} - \lambda (I\vec{v}) = 0$$

or $$(A - \lambda I)\vec{v} = 0$$

In our case

$$\left(\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \neq \vec{0}.$$

This implies that $(A - \lambda I)$ is not invertible.

then, $\det(A - \lambda I) = 0$

which implies $$\begin{vmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

or $(3-\lambda)(2-\lambda) - 2 = 0 \rightarrow \lambda^2 - 5\lambda + 6 - 2 = 0$

$\lambda^2 - 5\lambda + 4 = 0$ or $(\lambda - 4)(\lambda - 1) = 0$

Therefore, Two eigenvalues $\boxed{\lambda_1 = 4,\ \lambda_2 = 1}$

Find the eigenvalues of

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2(4-\lambda)+2$$

or $(1-\lambda)\left(\left((1-\lambda)(4-\lambda)+2\right)=0$

or $(1-\lambda)\left(\lambda^2-5\lambda+\underbrace{4+2}_{6}\right)=0$

Then $(1-\lambda)(\lambda-3)(\lambda-2)=0$

$\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$

Exercise

Show that $\lambda = 3$ is an eigenvalue of

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$ by finding a corresponding eigenvector.

If $\lambda = 3$ is eigenvalue of A, then there is $\vec{x} \neq \vec{0}$

$A\vec{x} = 3\vec{x}$

or

$$(A - 3I)\vec{x} = 0 \iff \left[\begin{array}{ccc|c} 4-3 & 0 & 1 & 0 \\ -2 & 1-3 & 0 & 0 \\ -2 & 0 & 1-3 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -2 & -2 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$-2x_2 = -2x_3 \Rightarrow \boxed{x_2 = x_3}$$

$$\boxed{x_1 = -x_3}$$

then,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

eigenvectors with eigenvalue $\underline{\lambda = 3}$

In fact,

$$A \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4+1 \\ 2+1 \\ 2+1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Strategy to find the eigenvalues of A and their Corresponding eigenvectors.

1) First, find λ's
2) Then use each λ to obtain its corresponding eigenvectors. $(\vec{x} \neq \vec{0})$.

1) Finding λ

Since $A\vec{x} = \lambda\vec{x}$, then

$A\vec{x} - \lambda\vec{x} = \vec{0}$ which is equivalent to

$A\vec{x} - \lambda I\vec{x} = (A - \lambda I)\vec{x} = 0$

where I is the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

We will require an eigenvector $\vec{x}$ of the eigenvalue λ to be $\vec{x} \neq \vec{0}$. Otherwise, any λ will work to satisfy $A\overset{=\vec{0}}{\vec{x}} = \lambda\vec{x} = \vec{0} = \vec{0}$.

So, λ should be chosen such that the Linear System

$$\boxed{(A - \lambda I)\vec{x} = \vec{0}}$$

has nontrivial solutions $\vec{x} \neq \vec{0}$.

Recall from the invertibility theorem that

A matrix B is invertible if and only if $B\vec{x}=\vec{0}$ has only the trivial solution.

So, $(A-\lambda I)\vec{x}=\vec{0}$ has nontrivial solutions if and only if $(A-\lambda I)$ is not invertible. which is equivalent to $|A-\lambda I|=0$

So, in our case $A_{2\times 2}$

$$A-\lambda I = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{bmatrix}$$

is not invertible if and only if

$$\boxed{\det(A-\lambda I) = (3-\lambda)(2-\lambda)-2=0.}$$

or $\lambda^2-5\lambda+6-2=0$

$$\boxed{\lambda^2-5\lambda+4=0} \Leftrightarrow (\lambda-4)(\lambda-1)=0$$

then $\boxed{\lambda_1=1}$ and $\boxed{\lambda_2=4}$

2) a) Finding eigenvectors corresponding to the eigenvalue $\underline{\underline{\lambda_1 = 1}}$.

$(A - 1I)\vec{x} = \vec{0}$ equiv. to

$$\begin{bmatrix} 3-1 & 1 \\ 2 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The problem is reduced to find the nontrivial soln. of this homog. syst. or the null Space of $A - 1I$.

Augmented matrix

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

then $2x_1 = (-1)x_2 \Rightarrow x_1 = (-1/2)x_2$ → Free Variable

Then, eigenvectors

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1/2\, x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

multiplying by 2.

or $\boxed{\vec{x} = t \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad t \text{ in } \mathbb{R},\ t \neq 0.}$

→ All possible eigenvectors corresponding to the eigenvalue $\lambda_1 = 1$.

2) b) Finding eigenvectors Corresponding to the eigenvalue $\underline{\underline{\lambda_2 = 4}}$.

$(A-4I)\vec{x} = \vec{0}$ equiv. to

$$\begin{bmatrix} 3-4 & 1 \\ 2 & 2-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

augmented matrix

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 2 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$-x_1 = -x_2$ or $\boxed{x_1 = x_2}$

then, eigenvectors

$$\vec{x} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

or $\vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, t in $\mathbb{R}$, $t \neq 0$

All possible eigenvectors corresponding to $\lambda_2 = 4$.

Eigenspaces of the eigenvalues λ's.

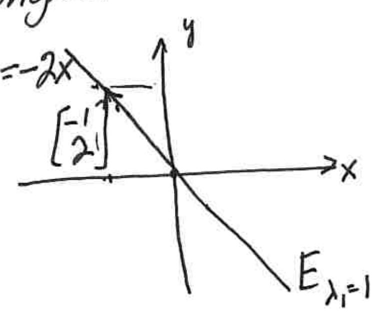
The Subspace

$$E_{\lambda_1=1} = \left\{ \vec{x} : \ \vec{x} = t\begin{bmatrix}-1\\2\end{bmatrix},\ t\in\mathbb{R}\ (\text{including } t=0) \right\}$$

is called the eigenspace of $\lambda_1 = 1$.

This subspace is the line with direction vector $\vec{d} = \begin{bmatrix}-1\\2\end{bmatrix}$ passing through the origin.

$y=-2x$



Also,

$$E_{\lambda_2=4} = \left\{ \vec{x} : \ \vec{x} = t\begin{bmatrix}1\\1\end{bmatrix},\ t\in\mathbb{R} \right\}$$

is called the eigenspace of $\lambda_2 = 4$.

This subspace is the line with direction vector $\vec{d} = \begin{bmatrix}1\\1\end{bmatrix}$ passing through the origin, or $y=x$